JOURNAL OF APPROXIMATION THEORY 21, 293–302 (1977)

The Operator of Surface Interpolation

F.-J. DELVOS AND W. SCHÄFER

Lehrstuhl für Mathematik I, Universität Siegen, D-5900 Siegen 21, West Germany

Communicated by Arthur Sard

Received March 22, 1976

INTRODUCTION

An abstract approach to the theory of spline interpolation has been developed by several authors [1, 7, 8, 13, 18]. In particular Sard's theory of optimal interpolation has been used to give a unified treatment of bivariate interpolation methods [5, 6, 11] including the surface interpolation schemes of Gordon [10].

In connection with an error analysis for abstract splines, an extension of Sard's method has been considered which is characterized by an unbounded hermitian operator associated with the quadratic functional of optimal interpolation [4, 9].

In this paper the operator of abstract surface interpolation [5] will be constructed. As an application, the hermitian operator associated with Mangeron's equation [3] will be studied. Finally, Green's function of Mangeron's operator is used to derive a representation formula for bivariate splines with arbitrarily distributed interpolation points.

1. THE OPERATOR OF OPTIMAL INTERPOLATION

An optimal interpolation method is conveniently described by the notion of a Sard system [8]

$$(X, Y, Z; U, F). \tag{1.1}$$

Here X, Y, Z are complex separable Hilbert spaces, and

$$U: X \to Y, \qquad F: X \to Z$$

are continuous linear mappings such that the completeness condition holds, i.e., the scalar product in X may be written in the form

$$((x, y)) = (Ux, Uy) + (Fx, Fy).$$

ISSN 0021-9045

The orthogonal projector P of $(X; ((\cdot,)))$ defined by

$$\text{Ker } P = \text{Ker } F \tag{1.2}$$

is called the spline projector of (1.1) and solves the optimal interpolation problem

$$\begin{gathered} s = Px \Leftrightarrow, \\ Fs = Fx, \qquad \| Us \| = \min_{Fx=Fy} \| Uy \|. \end{gathered} \tag{1.3}$$

In [9] a special class of interpolation processes has been considered:

The Sard system $(X, Y, Z; U, F)$ is called an *extended Sard system* iff Ker F is a dense and continuously imbedded linear subspace of Y:

$$\begin{gathered} \overline{\text{Ker } F} = Y, \\ \| x \| \leqslant c_0 \| Ux \| \qquad (x \in \text{Ker } F;\ c_0 > 0). \end{gathered} \tag{1.4}$$

Here and elsewhere $\| x \|$ denotes $\| x \|_Y$ $(x \in \text{Ker } F)$.

Then the operator U_0 defined by

$$\text{Dom } U_0 = \text{Ker } F, \qquad U_0 x = Ux$$

is closed in Y, and we have [9]

THEOREM 1. *The operator $A = U_0{}^* U_0$ is the unique positive definite hermitian operator in Y satisfying*

$$\begin{gathered} \text{Dom } A \subset \text{Ker } F, \\ (Ux, Uy) = (x, Ay) \qquad (x \in \text{Ker } F,\ y \in \text{Dom } A). \end{gathered} \tag{1.5}$$

A is called the *energy operator of the extended Sard system* $(X, Y, Z; U, F)$. In the terminology of Mikhlin [16, p. 17 ff]

$$H(x) = (Ux, Ux) \qquad (x \in X)$$

is the quadratic functional associated with A since

$$H(x) = (x, Ax)$$

where $x \in \text{Dom } A$. Mikhlin calls $z \in X$ a *weak solution* of the "boundary value" problem

$$Az = 0, \qquad Fz = Fx \qquad (x \in X)$$

iff

$$Fx = Fz, \qquad H(z) = \min_{Fy=0} H(x - y). \tag{1.6}$$

Mikhlin uses this variational concept of weak solution in the treatment of the classical Dirichlet problem (cf. [16, p. 19]). Here we go in the reverse direction: the weak solution is our starting point.

THEOREM 2. *For any $x \in X$ the optimal interpolant s of x is the unique weak solution of*

$$As = 0, \qquad Fs = Fx. \tag{1.7}$$

Proof. This follows immediately from (1.3).

Thus the concept of weak solvability is equivalent to the property of optimal interpolation. For a "spline" treatment of harmonic functions see, for instance, [8, 18].

2. ABSTRACT SURFACE INTERPOLATION

The method of abstract surface interpolation is based on two systems of optimal interpolation [5]

$$(X_1, Y_1, Z_1; U_1, F_1) \qquad (X_2, Y_2, Z_2; U_2, F_2) \tag{2.1}$$

and is characterized by the tuple

$$(X_1 \otimes X_2, Y_1 \otimes Y_2, Z_1 \otimes X_2 \times X_1 \otimes Z_2; U_1 \otimes U_2, F_1 \otimes I_2 \times I_1 \otimes F_2). \tag{2.2}$$

Here I_1 and I_2 are the identity mappings of X_1 and X_2 and P_1 and P_2 denote the spline projectors of (2.1). The following Theorem 3 has been proved in [5]:

THEOREM 3. *Suppose that F_1 and F_2 are normally solvable. Then*

$$B = P_1 \otimes I_2 + I_1 \otimes P_2 - P_1 \otimes P_2 \tag{2.3}$$

is the spline projector of surface interpolation (2.2).

An operator is called normally solvable if its range is closed. In this section we will construct the operator of surface interpolation. First we need

LEMMA 1. *The relation*

$$\mathrm{Ker}(F_1 \otimes I_2 \times I_1 \otimes F_2) = \mathrm{Ker}\, F_1 \otimes \mathrm{Ker}\, F_2 \tag{2.4}$$

is true.

Proof. The definition of the spline projector implies

$$\operatorname{Ker} F_1 = \operatorname{Ker} P_1 = \operatorname{Im}(I_1 - P_1),$$
$$\operatorname{Ker} F_2 = \operatorname{Ker} P_2 = \operatorname{Im}(I_2 - P_2),$$
$$\operatorname{Ker} (F_1 \otimes I_2 \times I_1 \otimes F_2) = \operatorname{Ker} B.$$

Since

$$I_1 \otimes I_2 - B = (I_1 - P_1) \otimes (I_2 - P_2), \tag{2.5}$$

the standard rules of tensor product [2] imply

$$\begin{aligned} &\operatorname{Ker} (F_1 \otimes I_2 \times I_1 \otimes F_2) \\ &= \operatorname{Im} (I_1 \otimes I_2 - B) \\ &= \operatorname{Im} (I_1 - P_1) \otimes \operatorname{Im} (I_2 - P_2) \\ &= \operatorname{Ker} F_1 \otimes \operatorname{Ker} F_2 . \end{aligned}$$

LEMMA 2. *Suppose that*

$$(X_1 , Y_1 , Z_1 ; U_1 , F_1), \qquad (X_2 , Y_2 , Z_2 ; U_2 , F_2)$$

are extended Sard systems. Then

$$(X_1 \otimes X_2 , Y_1 \otimes Y_2 , Z_1 \otimes X_2 \times X_1 \otimes Z_2 ; U_1 \otimes U_2 , F_1 \otimes I_2 \times I_1 \otimes F_2)$$

represents an extended Sard system, too.

Proof. Note first that

$$\overline{\operatorname{Ker} F_1} = Y_1 , \qquad \overline{\operatorname{Ker} F_2} = Y_2 .$$

Thus Lemma 1 implies

$$\overline{\operatorname{Ker}(F_1 \otimes I_2 \times I_1 \otimes F_2)} = Y_1 \otimes Y_2 .$$

From

$$\| x_1 \| \leqslant c_1 \| U_1 x_1 \| \qquad (x_1 \in \operatorname{Ker} F_1),$$
$$\| x_2 \| \leqslant c_2 \| U_2 x_2 \| \qquad (x_2 \in \operatorname{Ker} F_2),$$

we can conclude

$$\| x \| \leqslant c_1 c_2 \| U_1 \otimes U_2(x) \| \ (x \in \operatorname{Ker} (F_1 \otimes I_2 \times I_1 \otimes F_2)),$$

whence Lemma 2 is proved.

Let A_1 and A_2 be the energy operators of the systems (2.1). Then

$$G_1 = A_1^{-1}, \qquad G_2 = A_2^{-1} \tag{2.6}$$

are bounded positive hermitian operators in Y_1 and Y_2 (cf. [21, p. 482]). These operators satisfy

$$\begin{aligned} (x_1, y_1) &= (U_1 x_1, U_1 G_1 y_1) \qquad (x_1 \in \operatorname{Ker} F_1, y_1 \in Y_1), \\ (x_2, y_2) &= (U_2 x_2, U_2 G_2 y_2) \qquad (x_2 \in \operatorname{Ker} F_2, y_2 \in Y_2). \end{aligned} \tag{2.7}$$

Conversely, G_1 and G_2 are uniquely characterized by (2.7) and can be used to construct A_1 and A_2 [16].

If Dom A_1 and Dom A_2 are equipped with the scalar products

$$\begin{aligned} (x_1, y_1) &= (A_1 x_1, A_1 y_1), \\ (x_2, y_2) &= (A_2 x_2, A_2 y_2), \end{aligned}$$

they become Hilbert spaces, since A_1 and A_2 are closed operators. Then G_1 and G_2 constitute toplinear isomorphisms having the continuous inverses

$$\begin{aligned} A_1&\colon \operatorname{Dom} A_1 \to Y_1, \\ A_2&\colon \operatorname{Dom} A_2 \to Y_2. \end{aligned} \tag{2.8}$$

Therefore $A_1 \otimes A_2$ is well defined relative to the new topologies, and we can state our main result.

THEOREM 4. *The operator A defined by*

$$\begin{aligned} \operatorname{Dom} A &= \operatorname{Dom} A_1 \otimes \operatorname{Dom} A_2, \\ Ax &= A_1 \otimes A_2(x) \end{aligned} \tag{2.9}$$

is the energy operator of surface interpolation for (2.2).

Proof. Note first that Lemma 2 implies the existence of the operator of surface interpolation. Now

$$G = G_1 \otimes G_2 \tag{2.10}$$

is a bounded hermitian operator in $Y_1 \otimes Y_2$ which satisfies

$$(x_1 \otimes x_2, y_1 \otimes y_2) = (U_1 \otimes U_2(x_1 \otimes x_2), U_1 \otimes U_2(G(y_1 \otimes y_2)))$$
$$(x_1 \in \operatorname{Ker} F_1, y_1 \in Y_1 ; x_2 \in \operatorname{Ker} F_2, y_2 \in Y_2).$$

Taking into account Lemma 1 and the definition of tensor product [2] we obtain

$$(x, y) = (U_1 \otimes U_2(x), U_1 \otimes U_2(Gy))$$
$$(x \in \operatorname{Ker}(F_1 \otimes I_2 \times I_1 \otimes F_2), y \in Y_1 \otimes Y_2),$$

whence

$$A = G^{-1} = G_1^{-1} \otimes G_2^{-1} = A_1 \otimes A_2$$

follows. Thus Theorem 3 is proved.

Now an application of Theorem 2 yields

THEOREM 5. *For any $x \in X_1 \otimes X_2$ the surface interpolant*

$$s = Bx = (P_1 \otimes I_2 + I_1 \otimes P_2 - P_1 \otimes P_2)(x)$$

satisfies

$$A_1 \otimes A_2(s) = 0, \qquad F_1 \otimes I_2(s) = F_1 \otimes I_2(x), \qquad I_1 \otimes F_2(s) = I_1 \otimes F_2(x)$$

in the weak sense.

3. Mangeron's Operator

As an application we will apply the preceding results to blended linear interpolation [3, 5]. Blended linear interpolation is based on linear interpolation which is described by the extended Sard system

$$(W^1(J), L_2(J), \mathbb{C}^2; D, \epsilon_0 \times \epsilon_1). \tag{3.1}$$

Here $W^k(J)$ ($J = [0, 1]$) denotes the Sobolev space of function $f \in C^{k-1}(J)$ with $D^k f \in L_2(J)$; and ϵ_0, ϵ_1 are Dirac measures at the points 0, 1.

The operator A of linear interpolation is given by

$$Af = -D^2 f,$$
$$\text{Dom } A = \{f \in W^2(J) : f(0) = f(1) = 0\},$$

and the linear interpolant

$$\xi(s) = f(0)(1 - s) + f(1)s$$

is the unique solution of the boundary value problem

$$-D^2(\xi) = 0, \qquad \xi(0) = f(0), \qquad \xi(1) = f(1).$$

Let $R = J \times J$ denote the unit square and $R^\cdot = \partial R$ the boundary of R. Further we use the notations

$$\begin{aligned} W^{k,k}(R) &= W^k(J) \otimes W^k(J), \\ L_2(R) &= L_2(J) \otimes L_2(J), \\ D_x^k D_y^k &= D^k \otimes D^k, \\ f|_{R^\cdot} &= (f(\cdot, 0), f(\cdot, 1), f(0, \cdot), f(1, \cdot)). \end{aligned}$$

Then the tuple

$$(W^{1,1}(R), L_2(R), W^1(J)^4; D_x D_y, |_{R^\cdot})$$

represents the (extended) Sard system of blended linear interpolation, and an application of Theorem 4 yields

THEOREM 6. *The operator A of blended linear interpolation is Mangeron's operator:*

$$\begin{aligned} \operatorname{Dom} A &= \{f \in W^{2,2}(R) : f|_{R^\cdot} = 0\}, \\ Af &= D_x^2 D_y^2(f). \end{aligned} \tag{3.2}$$

From Theorem 5 we obtain for the case of blended linear interpolation

THEOREM 7. *The blended linear interpolant of $f \in W^{1,1}(R)$*

$$\begin{aligned} \xi(s, t) = {} & f(0, 1)(1 - s) + f(1, t)s \\ & + f(s, 0)(1 - t) + f(s, 1)t \\ & - f(0, 0)(1 - s)(1 - t) - f(0. 1)(1 - s)t \\ & - f(1, 0)\, s(1 - t) - f(1, 1)\, st \end{aligned}$$

is the unique weak solution of the Dirichlet problem for Mangeron's equation

$$D_x^2 D_y^2 \xi = 0, \qquad \xi|_{R^\cdot} = f|_{R^\cdot}. \tag{3.3}$$

Remark. It is well known that for a sufficiently smooth function $f \in C^{2,2}(R)$ the blended linear interpolant satisfies (3.3) in the usual sense (see [3, 10] for further generalizations).

4. INTERPOLATION WITH GREEN'S FUNCTION

It is well known in the classical theory of splines that Green's functions can be used to obtain representation formulas for splines [12]. It has been indicated in [4, 9] that interpolation with Green's functions is possible in a more abstract setting.

Suppose that B is a compact subset of $\mathbb{R}^m$. Then $C(B)$ denotes the Banach space of complex-valued continuous functions equipped with maximum norm topology

$$\|f\|_\infty = \max_{s \in B} |f(s)|.$$

It is further assumed that $\operatorname{Ker} F$ is a continuously imbedded subspace of $C(B)$, i.e., there is a positive constant M such that

$$\|f\|_\infty \leqslant M \,\|Uf\|$$

for any $f \in \operatorname{Ker} F$. Then $\operatorname{Ker} F$ possesses a reproducing kernel $K(s, t)$ [15], i.e., the relation

$$f(t) = (Uf, UK(\cdot, t)) \tag{4.1}$$

holds for any $t \in B$ and $f \in \operatorname{Ker} F$. For smoother functions $f \in \operatorname{Dom} A$ we have [4, 9]:

$$f(t) = (Af, K(\cdot, t)).$$

Therefore K is the Green's function of A and the kernel of G:

$$Gh(t) = (h, K(\cdot, t)) \qquad (t \in B;\ h \in Y). \tag{4.2}$$

Let $t_1, ..., t_m \in B$ be m distinct points such that the Dirac measures $\epsilon_{t_1}, ..., \epsilon_{t_m}$ are linear independent. Since $K(\cdot, t_1), ..., K(\cdot, t_m)$ are the representers of $\epsilon_{t_1}, ..., \epsilon_{t_m}$ the following minimum norm problem is solvable (cf. [14, p. 65; 15, p. 114 ff]):

THEOREM 8. *Let $c_1, ..., c_m$ be complex numbers. Among all functions $f \in \operatorname{Ker} F$ satisfying*

$$f(t_i) = c_i \qquad (i = 1, ..., m)$$

let η have the minimum norm

$$\| U\eta \| \leq \| Uf \|.$$

Then

$$\eta(s) = \sum_{i=1}^{m} b_i K(s, t_i) \tag{4.3}$$

where the coefficients satisfy

$$\sum_{i=1}^{m} K(t_j, t_i)\, b_i = c_j \qquad (j = 1, ..., m).$$

Let us first consider linear interpolation. In this case the Green's function

$$K(s, t) = s(1 - t) - (s - t)_+ \tag{4.4}$$

of the boundary value problem

$$-D^2 f = g, \qquad f(0) = f(1) = 0$$

is the reproducing kernel of

$$\operatorname{Ker}(\epsilon_0 \times \epsilon_1) = \{f \in W^1(J) : f(0) = f(1) = 0\}.$$

The real numbers $t_1, \ldots, t_m$ are supposed to satisfy

$$0 < t_1 < t_2 < \cdots < t_m < 1.$$

Then for any $f \in W^1(J)$ with $f(0) = f(1) = 0$ its optimal piecewise linear interpolant η has the unique representation

$$\eta(s) = \sum_{i=1}^{m} b_i(s(1 - t_i) - (s - t_i)_+).$$

Let us treat now blended linear interpolation. It follows from (4.2) and (2.10) that the Green's function of Mangeron's operator

$$D_x^2 D_y^2 f = g,$$
$$f|_{R^\cdot} = 0$$

has the form

$$K((s, t), (u, v)) = (s(1 - u) - (s - u)_+)(t(1 - v) - (t - v)_+). \quad (4.5)$$

Now an application of Theorem 8 yields an interpolation process with the aid of Green's function for Mangeron's equation.

THEOREM 9. *Let $(s_1, t_1), \ldots, (s_m, t_m) \in \mathring{R}$ be m distinct points. Further let $c_1, \ldots, c_m$ be complex numbers. Among all functions $f \in W^{1,1}(R)$ satisfying*

$$f(s_i, t_i) = c_i \qquad (i = 1, \ldots, m),$$
$$f|_{R^\cdot} = 0, \quad (4.6)$$

let η have the minimum norm

$$\iint_R |D_x D_y \eta(s, t)|^2 \, ds \, dt \leqslant \iint_R |D_x D_y f(s, t)|^2 \, ds \, dt.$$

Then

$$\eta(s, t) = \sum_{i=1}^{m} b_i[s(1 - s_i) - (s - s_i)_+][t(1 - t_i) - (t - t_i)_+]$$

where the coefficients satisfy

$$\sum_{i=1}^{m} b_i[s_j(1 - s_i) - (s_j - s_i)_+][t_j(1 - t_i) - (t_j - t_i)_+] = c_j$$
$$(j = 1, \ldots, m).$$

Remark. If the points $(s_1, t_1), \ldots, (s_m, t_m)$ form a rectangular mesh then $\eta(s, t)$ is a bilinear spline function [20].

Finally, it should be mentioned that the results stated for blended linear interpolation can be extended easily to the higher order blending methods as described in [3, 10].

REFERENCES

1. M. ATTEIA, "Étude de certains noyaux et théorie des fonctions "spline" en analyse numérique," Thèse, Université de Grenoble, June 1966.
2. J. M. BEREZANSKIJ, "Expansions in Eigenfunctions of Self-adjoint Operators," Amer. Math. Soc., Providence, R.I., 1968.
3. G. BIRKHOFF AND W. J. GORDON, The draftman's and related equations, *J. Approximation Theory* **1** (1968), 199–208.
4. F. -J. DELVOS, Optimale Interpolation mit der Methode von Ritz, *Z. Angew. Math. Mech.* **55** (1975), T234–T235.
5. F. -J. DELVOS, On surface interpolation, *J. Approximation Theory* **15** (1975), 209–213.
6. F. -J. DELVOS AND H. -W. KÖSTERS, On the variational characterization of bivariate interpolation methods, *Math. Z.* **145** (1975), 129–137.
7. F. -J. DELVOS AND W. SCHEMPP, On spline systems, *Monatsh. Math.* **74** (1970), 399–409.
8. F. -J. DELVOS AND W. SCHEMPP, Sard's method and the theory of spline systems, *J. Approximation Theory* **14** (1975), 230–243.
9. F. -J. DELVOS AND W. SCHEMPP, An extension of Sard's method, *in* "Spline Function, Karlsruhe 1975" (K. Böhmer, G. Meinardus, W. Schempp, Eds.), pp. 80–91, Lecture Notes in Mathematics, Vol. 501, Springer-Verlag, Berlin, 1976.
10. W. GORDON, Blending function methods of bivariate and multivariate interpolation and approximation, *SIAM J. Numer. Anal.* **8** (1971), 158–177.
11. W. HAUSSMANN, On multivariate spline systems, *J. Approximation Theory* **11** (1974), 285–305.
12. S. KARLIN, "Total Positivity," Stanford Univ. Press, Stanford, California, 1968.
13. P. -J. LAURENT, "Approximation et optimisation," Hermann, Paris, 1972.
14. D. G. LUENBERGER, "Optimization by Vector Space Methods," Wiley, New York, 1969.
15. H. MESCHKOWSKI, "Hilbertsche Räume mit Kernfunktion," Springer-Verlag, Berlin/Göttingen/Heidelberg, 1962.
16. S. G. MIKHLIN, "The Problem of the Minimum of a Quadratic Functional," Holden-Day, San Francisco/London/Amsterdam, 1965.
17. A. SARD, Optimal approximation, *J. Functional Analysis* **1** (1967), 222–244; **2** (1968), 368–369.
18. A. SARD, Approximation based on nonscalar observations, *J. Approximation Theory* **8** (1973), 3115–334.
19. W. SCHÄFER AND W. SCHEMPP, Splineapproximation in intermediären Räumen, *in* "Spline Functions, Karlsruhe 1975" (K. Böhmer, G. Meinardus, W. Schempp, Eds.), pp. 226–246, Lecture Notes in Mathematics, Vol. 501, Springer-Verlag, Berlin, 1976.
20. M. H. SCHULTZ, "Spline Analysis," Prentice-Hall, Englewood Cliffs, N. J., 1973.
21. V. I. SMIRNOW, "Lehrgang der höheren Mathematik, Teil V," VEB Deutscher Verlag der Wissenschaften, Berlin, 1967.